

SEPARATION AXIOMS ON BI-GENERALIZED TOPOLOGICAL SPACES

A. DEB RAY* AND RAKESH BHOWMICK**

ABSTRACT. In this paper, introducing various separation axioms on a bi-GTS, it has been observed that such separation axioms actually unify the well-known separation axioms on topological spaces. Several characterizations of such separation properties of a bi-GTS are established in terms of γ_{μ_i, μ_j} -closure operator, generalized cluster sets of functions and graph of functions.

1. Introduction and preliminaries

The concept of bi-Generalized topology (in short, bi-GTS) was introduced by Á. Császár and E. Makai Jr. in [5]. We study certain separation axioms on bi-GTS and find their characterizations in terms of γ_{μ_i, μ_j} -closure operator [5], graph of a function and generalized cluster sets [2] of a function. It is worth noting that the well-known separation axioms of bi-topological and hence topological spaces, follow as special cases for suitable choices of the bi-GTs.

In the next section, we investigate the behaviour of a bi-GTS obeying separation properties, in terms of a generalized closure operator called γ_{μ_i, μ_j} -closure operator [5]; while in the last section, a bi-GTS under separation properties are discussed in the light of graph of a function and generalized cluster sets [2] of a function.

We now state certain useful definitions and quote several existing results that we require in the next two sections.

DEFINITION 1.1. ([4]) Let X be a nonempty set and μ be a collection of subsets of X (i.e. $\mu \subseteq \mathcal{P}(X)$). μ is called a generalized topology

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Correspondence should be addressed to A. Deb Ray, atashi@hotmail.com.

(briefly GT) on X iff $\emptyset \in \mu$ and $G_\lambda \in \mu$ for $\lambda \in \Lambda (\neq \emptyset)$ implies $\cup_{\lambda \in \Lambda} G_\lambda \in \mu$. The pair (X, μ) is called a generalized topological space (briefly GTS). The elements of μ are called μ -open sets and their complements are called μ -closed sets. The generalized closure of a subset S of X , denoted by $c_\mu(S)$, is the intersection of all μ -closed sets containing S . The set of all μ -open sets containing an element $x \in X$ is denoted by $\mu(x)$.

For a topological space (X, τ) , set of all open, δ -open [18], semi open [10] and pre open [11] subsets of X are denoted respectively by $\tau(X)$, $\Delta(X)$, $SO(X)$ and $PO(X)$.

Let μ_1, μ_2 be two GTs on a non-empty set X . Then (X, μ_1, μ_2) is called bi-Generalized topological space (briefly bi-GTS).

DEFINITION 1.2. ([5]) On a bi-GTS (X, μ_1, μ_2) , $\gamma_{\mu_i, \mu_j} : P(X) \rightarrow P(X)$ is defined by

$$\gamma_{\mu_i, \mu_j}(A) = \{x \in X : c_{\mu_j}M \cap A \neq \phi \text{ for all } M \in \mu_i(x)\},$$

for each $A \subseteq X, i, j = 1, 2 (i \neq j)$. $\theta(\mu_i, \mu_j), \delta(\mu_i, \mu_j) \subseteq P(X)$, defined respectively by

$$\theta(\mu_i, \mu_j) = \{A \subseteq X : \text{for each } x \in A \text{ there exists } M \in \mu_i(x) \text{ such that } c_{\mu_j}M \subseteq A\}, i, j = 1, 2 (i \neq j),$$

and

$$\delta(\mu_i, \mu_j) = \{A \subseteq X : \text{for each } x \in A \exists \mu_j - \text{closed set } Q \text{ with } x \in i_{\mu_i}Q \subseteq A\}, i, j = 1, 2 (i \neq j),$$

also form GTs on X . The elements of $\theta(\mu_i, \mu_j)$ (resp. $\delta(\mu_i, \mu_j)$) are called $\theta(\mu_i, \mu_j)$ (resp. $\delta(\mu_i, \mu_j)$)-open and the complements are called $\theta(\mu_i, \mu_j)$ (resp. $\delta(\mu_i, \mu_j)$)-closed.

THEOREM 1.3. ([5]) Let (X, μ_1, μ_2) be a bi-GTS and $A \subseteq X$. Then the following hold:

- (1) $\theta(\mu_i, \mu_j) \subseteq \delta(\mu_i, \mu_j) \subseteq \mu_i$.
- (2) $A \subseteq \gamma_{\mu_i, \mu_j}(A) \subseteq c_{\theta(\mu_i, \mu_j)}(A)$.
- (3) A is $\theta(\mu_i, \mu_j)$ -closed iff $A = \gamma_{\mu_i, \mu_j}(A)$.

THEOREM 1.4. ([14]) Let (X, μ_1, μ_2) be a bi-GTS. Then for any μ_j -open set A we have $\gamma_{\mu_i, \mu_j}(A) = c_{\mu_i}(A)$.

THEOREM 1.5. For any subset A in a bi-GTS (X, μ_1, μ_2) , $\gamma_{\mu_i, \mu_j}(A) = \cap \{c_{\mu_i}V : A \subseteq V \in \mu_j\}$.

Let μ_1, μ_2 be two GTs on a non-empty set X and $A \subseteq X$. A is said to be $r(\mu_i, \mu_j)$ -open (resp. $r(\mu_i, \mu_j)$ -closed) [5] if $A = i_{\mu_i}(c_{\mu_j}(A))$ (resp. $A = c_{\mu_i}(i_{\mu_j}(A))$).

THEOREM 1.6. ([5]) $x \in c_{\delta(\mu_i, \mu_j)}A$ iff $A \cap R \neq \phi$ for every $r(\mu_i, \mu_j)$ -open set R containing x .

Let (X, μ_1, μ_2) and (Y, η_1, η_2) be two bi-GTS. The GT $\nu_i (i = 1, 2)$ on the cartesian product $X \times Y$ is defined by $\nu_i = \mu_i \times \eta_j$ for $i, j = 1, 2 (i \neq j)$; Then $(X \times Y, \nu_1, \nu_2)$ is again a bi-GTS. Also, for the bi-GTS (X, μ_1, μ_2) , $(X \times X, \nu_1, \nu_2)$ is a bi-GTS where $\nu_i = \mu_i \times \mu_j$ for $i, j = 1, 2 (i \neq j)$.

2. Separation axioms in terms of γ_{μ_i, μ_j} -closure operator

In this section, we introduce different separation axioms on a bi-GTS and establish their interrelationships. Also, such separation axioms are characterized here using generalized closure operator, called γ_{μ_i, μ_j} -closure operator.

DEFINITION 2.1. Let μ be a GT on a non-empty set X . Then for any $A \subseteq X$, $k_\mu A = \cap \{U \in \mu : A \subseteq U\}$.

DEFINITION 2.2. Let (X, μ_1, μ_2) be a bi-GTS. Then for any point $x \in X$ we define $ij\text{-}ck(A) = (c_{\mu_i}A) \cap (k_{\mu_j}A)$; for $i, j = 1, 2 (i \neq j)$.

If $A = \{x\}$, we will write $ij\text{-}ck\{x\}$ for $ij\text{-}ck(\{x\})$.

LEMMA 2.3. Let x be an arbitrary point in a bi-GTS (X, μ_1, μ_2) . Then

- (1) $y \in ji\text{-}ck\{x\}$ iff $ij\text{-}ck\{x\} \subseteq ij\text{-}ck\{y\}$.
- (2) $c_{\mu_i}(ij\text{-}ck\{x\}) = c_{\mu_i}\{x\}$.
- (3) $k_{\mu_j}(ij\text{-}ck\{x\}) = k_{\mu_j}\{x\}$.
- (4) $\gamma_{\mu_i, \mu_j}(ij\text{-}ck\{x\}) = \gamma_{\mu_i, \mu_j}\{x\}$.
- (5) for any μ_j -open set U containing x , $k_{\mu_j}\{x\} \subseteq U$.
- (6) for any μ_i -closed set F containing x , $c_{\mu_i}(ij\text{-}ck\{x\}) \subseteq F$.
- (7) $k_{\mu_i}(k_{\mu_i}A) = k_{\mu_i}A$ for $A \subseteq X$.
- (8) $\gamma_{\mu_i, \mu_j}(k_{\mu_j}A) = \gamma_{\mu_i, \mu_j}A$ for $A \subseteq X; i, j = 1, 2 (i \neq j)$.

Proof.

- (1) Let, $y \in ji\text{-}ck\{x\}$. Suppose $z \in ij\text{-}ck\{x\}$. Now $y \in ji\text{-}ck\{x\}$ implies $y \in c_{\mu_j}\{x\}$, $y \in k_{\mu_i}\{x\}$ and $z \in ij\text{-}ck\{x\}$ implies $z \in c_{\mu_i}\{x\}$, $z \in k_{\mu_j}\{x\}$. Again $z \in c_{\mu_i}\{x\}$ and $y \in k_{\mu_i}\{x\}$ together imply $z \in c_{\mu_i}\{y\}$. Also $y \in c_{\mu_j}\{x\}$ and $z \in k_{\mu_j}\{x\}$ together imply $z \in k_{\mu_j}\{y\}$. So, $z \in c_{\mu_i}\{y\} \cap k_{\mu_j}\{y\} = ij\text{-}ck\{y\}$. Hence $ij\text{-}ck\{x\} \subseteq ij\text{-}ck\{y\}$.
 Conversely, let $ij\text{-}ck\{x\} \subseteq ij\text{-}ck\{y\}$. Since, $x \in ij\text{-}ck\{x\} \subseteq ij\text{-}ck\{y\}$, So

- $x \in c_{\mu_i}\{y\}$ and $x \in k_{\mu_j}\{y\}$. Now $x \in c_{\mu_i}\{y\}$ implies $y \in k_{\mu_i}\{x\}$. Also $x \in k_{\mu_j}\{y\}$ implies $y \in c_{\mu_j}\{x\}$. So, $y \in c_{\mu_j}\{x\} \cap k_{\mu_i}\{x\} = ij\text{-}ck\{x\}$.
- (2) Let $z \in c_{\mu_i}(ij\text{-}ck\{x\})$. Therefore for all $U \in \mu_i(z)$, $U \cap (ij\text{-}ck\{x\}) \neq \phi$ and so $U \cap (c_{\mu_i}\{x\}) \neq \phi$ i.e. $z \in c_{\mu_i}(c_{\mu_i}\{x\}) = c_{\mu_i}\{x\}$. Hence $c_{\mu_i}(ij\text{-}ck\{x\}) \subseteq c_{\mu_i}\{x\}$.
- Conversely, $\{x\} \subseteq ij\text{-}ck\{x\}$ implies $c_{\mu_i}\{x\} \subseteq c_{\mu_i}(ij\text{-}ck\{x\})$. Thus $c_{\mu_i}(ij\text{-}ck\{x\}) = c_{\mu_i}\{x\}$.
- (3) Let $y \in k_{\mu_j}(ij\text{-}ck\{x\})$ but $y \notin k_{\mu_j}\{x\}$. Then there exists $U \in \mu_j(x)$ such that $y \notin U$. Also, $y \in k_{\mu_j}(ij\text{-}ck\{x\}) \Rightarrow ij\text{-}ck\{x\} \cap c_{\mu_j}\{y\} \neq \phi \Rightarrow c_{\mu_j}\{y\} \cap k_{\mu_j}\{x\} \neq \phi$. Hence there exists $z \in c_{\mu_j}\{y\} \cap k_{\mu_j}\{x\}$. Then every μ_j -open neighbourhood of x contains y , a contradiction.
- (4) Let, $y \in \gamma_{\mu_i, \mu_j}(ij\text{-}ck\{x\})$ and if possible let $y \notin \gamma_{\mu_i, \mu_j}\{x\}$. Then there exists $U \in \mu_i(y)$ such that $x \notin c_{\mu_j}U$. Again $y \in \gamma_{\mu_i, \mu_j}(ij\text{-}ck\{x\})$ implies $c_{\mu_j}U \cap ij\text{-}ck\{x\} \neq \phi$ i.e. $c_{\mu_j}U \cap k_{\mu_j}\{x\} \neq \phi$ and so there exists $z \in c_{\mu_j}U \cap k_{\mu_j}\{x\}$. Again since, $x \in X \setminus c_{\mu_j}U \in \mu_j$ and $z \in k_{\mu_j}\{x\}$, so, $z \in X \setminus c_{\mu_j}U$, which is not possible. Hence, $\gamma_{\mu_i, \mu_j}(ij\text{-}ck\{x\}) \subseteq \gamma_{\mu_i, \mu_j}\{x\}$.
- Conversely, $x \in ij\text{-}ck\{x\}$ implies $\gamma_{\mu_i, \mu_j}\{x\} \subseteq \gamma_{\mu_i, \mu_j}(ij\text{-}ck\{x\})$.
- (5) Let, $z \in k_{\mu_j}\{x\}$ and $U \in \mu_j(x)$. Clearly $z \in U$. Thus $k_{\mu_j}\{x\} \subseteq U$.
- (6) By (2) $c_{\mu_i}(ij\text{-}ck\{x\}) = c_{\mu_i}\{x\}$ and cosequently $c_{\mu_i}(ij\text{-}ck\{x\}) \subseteq F$.
- (7) R.H.S \subseteq L.H.S. We now show that L.H.S \subseteq R.H.S. Let $y \notin R.H.S$. Then there exists a μ_i open set containing A s.t $y \notin U$. Again $A \subseteq U$ and $U \in \mu_i$ implies that $k_{\mu_i}\{A\} \subseteq U$ and consequently $y \notin R.H.S$.
- (8) $R.H.S \subseteq L.H.S$. We now show that L.H.S \subseteq R.H.S. Let $y \notin R.H.S$. Then there exists a μ_i open set U containing y s.t. $c_{\mu_j}U \cap A = \phi$, Consequently $c_{\mu_j}U \cap k_{\mu_j}\{A\} = \phi$ (Since, $k_{\mu_j}\{A\}$ is the intesection of all μ_j open set containing A). Hence $y \notin L.H.S$. \square

COROLLARY 2.4. For any point x in a bi-GTS (X, μ_1, μ_2) the following hold :

- (1) For any μ_j -open set U containing x , $ij\text{-}ck\{x\} \subseteq U$.
- (2) For any μ_i -closed set F containing x , $ij\text{-}ck\{x\} \subseteq F$.
- (3) For any point x , $ij\text{-}ck(\{ij\text{-}ck\{x\}) = ij\text{-}ck\{x\}$.

Proof.

- (1) Follows from (5) of Lemma 2.3 and definition of $ij\text{-}ck\{x\}$.
- (2) Follows from (6) of Lemma 2.3 and definition of $c_{\mu_i}\{x\}$.
- (3) Follows from (2) and (3) of Lemma 2.3. \square

DEFINITION 2.5. A bi-GTS (X, μ_1, μ_2) is said to be pairwise R_0 -space if for each μ_i -open set G and for each $x \in G$, $c_{\mu_j}\{x\} \subseteq G$; for $i, j = 1, 2$ ($i \neq j$).

μ_1	μ_2	pairwise R_0
τ	τ	R_0 [8]
$SO(X)$	$SO(X)$	semi R_0 [7]
$PO(X)$	$PO(X)$	pre R_0 [3]

THEOREM 2.6. *If (X, μ_1, μ_2) is pairwise R_0 , then for each $x \in X$, $\gamma_{\mu_j, \mu_i}\{x\} \setminus c_{\mu_i}\{x\}$ is a union of μ_j -closed sets; for $i, j = 1, 2$ ($i \neq j$).*

Proof. Let, $y \in \gamma_{\mu_j, \mu_i}\{x\} \setminus c_{\mu_i}\{x\}$. Then $y \in X \setminus c_{\mu_i}\{x\}$. Since X is pairwise R_0 , $c_{\mu_i}\{x\} \cap c_{\mu_j}\{y\} = \phi$. Now $y \in \gamma_{\mu_j, \mu_i}\{x\}$ implies $c_{\mu_j}\{y\} \subseteq \gamma_{\mu_j, \mu_i}\{x\}$. Thus $c_{\mu_j}\{y\} \subseteq \gamma_{\mu_j, \mu_i}\{x\} \setminus c_{\mu_i}\{x\}$. Consequently $\gamma_{\mu_j, \mu_i}\{x\} \setminus c_{\mu_i}\{x\}$ is a union of c_{μ_j} -closed sets. \square

THEOREM 2.7. *If for every pair of distinct point x, y in a bi-GTS (X, μ_1, μ_2) , either $c_{\mu_i}\{x\} = c_{\mu_j}\{y\}$ or $c_{\mu_i}\{x\} \cap c_{\mu_j}\{y\} = \phi$, for $i, j = 1, 2$ ($i \neq j$), then (X, μ_1, μ_2) is pairwise R_0 .*

Proof. Let G be a μ_i -open set containing $y \in X$. For any $x \in X \setminus G$ as $y \notin c_{\mu_i}\{x\}$, $c_{\mu_i}\{x\} \neq c_{\mu_j}\{y\}$. By the hypothesis, $c_{\mu_i}\{x\} \cap c_{\mu_j}\{y\} = \phi$ which gives $x \notin c_{\mu_j}\{y\}$; i.e. there exists $V_x \in \mu_j(x)$ such that $y \notin V_x$. Let $A = \cup\{V_x : x \in X \setminus G\}$. Then $y \notin A$ and $A \in \mu_j$. So $X \setminus A$ is a μ_j -closed set containing y . Also $X \setminus G \subseteq A$ i.e. $X \setminus A \subseteq G$. Therefore $c_{\mu_j}\{y\} \subseteq G$ and hence (X, μ_1, μ_2) is pairwise R_0 . \square

DEFINITION 2.8. A bi-GTS (X, μ_1, μ_2) is said to be pairwise R_1 if for any two points $x, y \in X$ such that $x \notin c_{\mu_i}\{y\}$, there are μ_i -open set U containing x and μ_j -open set V containing y such that $U \cap V = \phi$; where $i, j = 1, 2$ ($i \neq j$).

μ_1	μ_2	pairwise R_1
τ	τ	R_1 [8]
$SO(X)$	$SO(X)$	semi R_1 [7]
$PO(X)$	$PO(X)$	pre R_1 [3]

REMARK 2.9. Every pairwise R_1 space is pairwise R_0 .

Proof. Let (X, μ_1, μ_2) be pairwise R_1 . Let G be a μ_i open set and $x \in G$. If $X \setminus G = \phi$ then the proof is obvious. So let us consider the case $X \setminus G \neq \phi$ and $y \notin G$. Consequently $x \notin c_{\mu_i}\{y\}$. Since X is pairwise R_1 there exist $U_y \in \mu_i(x)$ and $V_y \in \mu_j(y)$ s.t. $U_y \cap V_y = \phi$. Let $V = \cup_{y \notin G} V_y$ and $F = X \setminus V$. Then F is a μ_j closed set containing x s.t. $F \subseteq G$ i.e. $c_{\mu_j}\{x\} \subseteq G$. Hence X is pairwise R_0 . \square

But the converse is not true. This follows from the following example.

EXAMPLE 2.10. Let us consider the set $X = \{a, b, c\}$. Let $\mu_1 = \mu_2 = \mu = \{\phi, \{a, b\}, \{b, c\}, \{c, a\}, X\}$. Then $a \notin c_{\mu_i}\{b\} = \{b\}$ but every μ -open set containing them intersect each other. i.e. X is not pairwise R_1 . Again for every μ_i -open set G and for each $x \in G$, $c_{\mu_j}\{x\} \subseteq G$, for $i, j = 1, 2$. i.e X is pairwise R_0 .

THEOREM 2.11. Let (X, μ_1, μ_2) be a bi-GTS. Then the following are equivalent:

- (a) X is pairwise R_1 .
- (b) $ij\text{-}ck\{x\} = \gamma_{\mu_i, \mu_j}\{x\}$, for each $x \in X$.
- (c) $ij\text{-}ck\{x\}$ is $\theta(\mu_i, \mu_j)$ -closed set, for each $x \in X$.
- (d) $\gamma_{\mu_i, \mu_j}\{x\} = c_{\mu_i}\{x\}$, for each $x \in X$.
- (e) $\gamma_{\mu_i, \mu_j}\{x\} = k_{\mu_j}\{x\}$, for each $x \in X$.
- (f) $c_{\mu_i}\{x\}$ is $\theta(\mu_i, \mu_j)$ -closed, for each $x \in X$.
- (g) $k_{\mu_j}\{x\}$ is $\theta(\mu_i, \mu_j)$ -closed, for each $x \in X$.
- (h) If F is μ_i -closed set containing x , then $\gamma_{\mu_i, \mu_j}\{x\} \subseteq F$, for each $x \in X$.
- (i) If U is a μ_j -open set containing x , then for each $x \in X$, $\gamma_{\mu_i, \mu_j}\{x\} \subseteq U$; $i, j = 1, 2$ ($i \neq j$).

Proof.

(a) \Rightarrow (b): Let $x \in X$. Also let $y \in X$ be such that $y \notin ij\text{-}ck\{x\}$, then $y \notin c_{\mu_i}\{x\} \cap k_{\mu_j}\{x\}$. Now if $y \notin k_{\mu_j}\{x\}$ then $x \notin c_{\mu_j}\{y\}$. since X is pairwise R_1 , there exist $U \in \mu_j(x)$ and $V \in \mu_i(y)$ such that $U \cap V = \phi$. Then $y \notin c_{\mu_i}\{x\}$. Thus $y \notin k_{\mu_j}\{x\}$ implies $y \notin c_{\mu_i}\{x\}$. If possible let $y \in \gamma_{\mu_i, \mu_j}\{x\}$, then for all μ_i -open set W containing y , $x \in c_{\mu_j}W$. Since $y \notin c_{\mu_i}\{x\}$ and X is pairwise R_1 there exist $W_1 \in \mu_i(y)$ and $W_2 \in \mu_j(x)$ such that $W_1 \cap W_2 = \phi$ i.e. $x \notin c_{\mu_j}W_1$, a contradiction. So $y \notin \gamma_{\mu_i, \mu_j}\{x\}$ and hence $\gamma_{\mu_i, \mu_j}\{x\} \subseteq ij\text{-}ck\{x\}$.

On the other hand if $y \in ij\text{-}ck\{x\}$, then $y \in c_{\mu_i}\{x\} \subseteq \gamma_{\mu_i, \mu_j}\{x\}$ so that $ij\text{-}ck\{x\} \subseteq \gamma_{\mu_i, \mu_j}\{x\}$.

(b) \Leftrightarrow (c): follows from lemma 2.3.

(b) \Rightarrow (d): This is evident from the fact that $ij\text{-}ck\{x\} \subseteq c_{\mu_i}\{x\} \subseteq \gamma_{\mu_i, \mu_j}\{x\}$, for each $x \in X$.

(d) \Rightarrow (a): Let $x, y \in X$ with $y \notin c_{\mu_i}\{x\} = \gamma_{\mu_i, \mu_j}\{x\}$. Then there exists $U \in \mu_i(y)$ such that $x \notin c_{\mu_j}U$. Hence $X \setminus c_{\mu_j}U$ is a μ_j -open set containing x such that $(X \setminus c_{\mu_j}U) \cap U = \phi$, proving that (X, μ_1, μ_2) is pairwise R_1 .

(d) \Rightarrow (e): Let $y \in \gamma_{\mu_i, \mu_j}\{x\} = c_{\mu_i}\{x\}$. If possible let $y \notin k_{\mu_j}\{x\}$. Then $x \notin c_{\mu_j}\{y\} = \gamma_{\mu_j, \mu_i}\{x\}$, a contradiction. Thus $\gamma_{\mu_i, \mu_j}\{x\} \subseteq k_{\mu_j}\{x\}$.

Conversely, if $y \notin \gamma_{\mu_i, \mu_j}\{x\}$, then there exists $W \in \mu_i(y)$ such that

$x \notin c_{\mu_j} W \supseteq c_{\mu_j} \{y\}$ and so $y \notin k_{\mu_j} \{x\}$. Thus $k_{\mu_j} \{x\} \subseteq \gamma_{\mu_i, \mu_j} \{x\}$ and hence $\gamma_{\mu_i, \mu_j} \{x\} = k_{\mu_j} \{x\}$.

(e) \Rightarrow (d): Let, $y \in \gamma_{\mu_i, \mu_j} \{x\} = k_{\mu_j} \{x\}$, then $x \in c_{\mu_j} \{y\}$. Now $y \notin c_{\mu_i} \{x\}$ implies $x \notin k_{\mu_i} \{y\} = \gamma_{\mu_j, \mu_i} \{y\} \supseteq c_{\mu_j} \{y\}$ which is a contradiction. Consequently, $\gamma_{\mu_i, \mu_j} \{x\} \subseteq c_{\mu_i} \{x\}$. The other part, i.e. $c_{\mu_i} \{x\} \subseteq \gamma_{\mu_i, \mu_j} \{x\}$ is obvious.

(a) \Rightarrow (f): Follows from (b), (c) and (d).

(f) \Rightarrow (d): $\{x\} \subseteq c_{\mu_i} \{x\}$ gives $\gamma_{\mu_i, \mu_j} \{x\} \subseteq \gamma_{\mu_i, \mu_j} (c_{\mu_i} \{x\}) = c_{\mu_i} \{x\}$ and $c_{\mu_i} \{x\} \subseteq \gamma_{\mu_i, \mu_j} \{x\}$ is obvious. Hence $\gamma_{\mu_i, \mu_j} \{x\} = c_{\mu_i} \{x\}$ for each $x \in X$.

(a) \Rightarrow (g): It follows from (b), (c) and (e).

(g) \Rightarrow (e): Since $k_{\mu_j} \{x\}$ is $\theta(\mu_i, \mu_j)$ -closed for each $x \in X$, $\gamma_{\mu_i, \mu_j} (k_{\mu_j} \{x\}) = k_{\mu_j} \{x\}$ and so (e) follows from Corollary 2.4.

(h) \Rightarrow (d): For each $x \in X$, $c_{\mu_i} \{x\}$ is a μ_i -closed set containing x and hence by (h), $\gamma_{\mu_i, \mu_j} \{x\} \subseteq c_{\mu_i} \{x\}$. Again since $c_{\mu_i} \{x\} \subseteq \gamma_{\mu_i, \mu_j} \{x\}$ is obvious, we have $c_{\mu_i} \{x\} = \gamma_{\mu_i, \mu_j} \{x\}$. The implications “(b) \Rightarrow (h)”, “(b) \Rightarrow (i)” and “(i) \Rightarrow (e)” follow respectively from (4.), (3.) and (2.) of corollary 2.4. \square

COROLLARY 2.12. *If (X, μ_1, μ_2) is pairwise R_1 -space, then $\gamma_{\mu_i, \mu_j} \{x\}$ is $\theta(\mu_i, \mu_j)$ -closed for each $x \in X$.*

DEFINITION 2.13. For any subset A of a bi-GTS (X, μ_1, μ_2) we define $\gamma'_{\mu_i, \mu_j} (A) = \{x \in X : \gamma_{\mu_i, \mu_j} \{x\} \cap A \neq \phi\}$; $i, j = 1, 2$ ($i \neq j$).

It is easy to observe from the above definition that for $A, B \subseteq X$,

- (i) $A \subseteq k_{\mu_i} A \subseteq \gamma'_{\mu_i, \mu_j} A$, (ii) $A \subseteq B \Rightarrow \gamma'_{\mu_i, \mu_j} A \subseteq \gamma'_{\mu_i, \mu_j} B$ and
- (iii) $\gamma'_{\mu_i, \mu_j} (A \cup B) = \gamma'_{\mu_i, \mu_j} A \cup \gamma'_{\mu_i, \mu_j} B$.

DEFINITION 2.14. ([2]) Let (X, μ_1, μ_2) be two bi-GTS. Then X is said to be pairwise-Hausdorff if for $x \neq y$ in X , there exist $U \in \mu_i(x)$, $V \in \mu_j(y)$ such that $U \cap V = \emptyset$. $i, j = 1, 2$ ($i \neq j$).

μ_1	μ_2	pairwise-Hausdorff
τ	τ	T_2
$SO(X)$	$SO(X)$	semi T_2 [12]
$PO(X)$	$PO(X)$	pre T_2 [9]

Every pairwise Hausdorff space is also a pairwise R_1 space. The example below shows that the converse is not necessarily true.

EXAMPLE 2.15. *Let us consider the set $X = \{a, b, c\}$. Let $\mu_1 = \mu_2 = \mu = \{\phi, \{a\}, \{b, c\}, X\}$. Then for the point b and c there exist no pair of disjoint μ -open containing them. i.e. X is not pairwise Hausdorff. But*

for any two points $x, y \in X$ s.t. $x \in c_{\mu_1}\{y\}$, there are μ_1 -open set U containing x and μ_2 -open set V containing y s.t. $U \cap V = \phi$. i.e. X is pairwise R_1 .

THEOREM 2.16. For any bi-GTS (X, μ_1, μ_2) the following are equivalent :

- (a) X is pairwise Hausdorff.
- (b) For each $x \in X$, $\{x\} = \gamma_{\mu_i, \mu_j}\{x\} \cup \gamma_{\mu_j, \mu_i}\{x\}$.
- (c) For any two distinct points x, y of X , $\gamma_{\mu_i, \mu_j}\{x\} \cap \gamma_{\mu_j, \mu_i}\{y\} = \phi$.
- (d) For any subset A of X , $A = \gamma'_{\mu_i, \mu_j}(A)$; $i, j = 1, 2$ ($i \neq j$).

Proof.

- (a) \Rightarrow (b): Let $y \in X$ such that $y \neq x$. Then there exist $U \in \mu_i(x)$ and $V \in \mu_j(y)$ such that $U \cap V = \phi$. Thus $x \notin c_{\mu_i}V$ and hence $y \notin \gamma_{\mu_j, \mu_i}\{x\}$. Similarly $y \notin \gamma_{\mu_i, \mu_j}\{x\}$. Consequently, $\{x\} = \gamma_{\mu_i, \mu_j}\{x\} \cup \gamma_{\mu_j, \mu_i}\{x\}$.
- (b) \Rightarrow (c): straightforward.
- (c) \Rightarrow (d): $A \subseteq \gamma'_{\mu_i, \mu_j}(A)$ is evident. Now let $x \in \gamma'_{\mu_i, \mu_j}(A)$ so that $\gamma_{\mu_i, \mu_j}\{x\} \cap A \neq \phi$. let $y \in X$ such that $y \neq x$. Then $\gamma_{\mu_i, \mu_j}\{y\} \cap \gamma_{\mu_i, \mu_j}\{x\} = \phi$ and consequently, $y \notin \gamma_{\mu_i, \mu_j}\{x\}$. Thus $x \in A$ and hence $\gamma'_{\mu_i, \mu_j}(A) \subseteq A$.
- (d) \Rightarrow (a): Let x and y be any two distinct points of X . Now, $\{x\} = \gamma'_{\mu_i, \mu_j}\{x\}$ implies $y \notin \gamma'_{\mu_i, \mu_j}\{x\}$ and hence $x \notin \gamma_{\mu_i, \mu_j}\{y\}$. So there exists a $U \in \mu_i(x)$ such that $y \notin c_{\mu_j}U$, i.e. $y \in (X \setminus c_{\mu_j}U) (= V, \text{ say}) \in \mu_j$ and $U \cap V = \phi$. Hence the bi-GTS is pairwise Hausdorff. \square

COROLLARY 2.17. The following statements are equivalent for a bi-GTS (X, μ_1, μ_2) :

- (a) X is pairwise Hausdorff.
- (b) For each $x \in X$, $\{x\} = \gamma_{\mu_1, \mu_2}\{x\}$, i.e. every singleton of X is $\theta(\mu_1, \mu_2)$ -closed.
- (c) For each $x \in X$, $\{x\} = \gamma_{\mu_2, \mu_1}\{x\}$, i.e. every singleton of X is $\theta(\mu_2, \mu_1)$ -closed.

DEFINITION 2.18. A bi-GTS (X, μ_1, μ_2) is said to be pairwise Urysohn if for any two distinct point x, y of X , there exist $U \in \mu_1(x)$ and $V \in \mu_2(x)$ such that $c_{\mu_2}U \cap c_{\mu_1}V = \phi$.

μ_1	μ_2	Pairwise Urysohn
τ	τ	Urysohn [19]
$SO(X)$	$SO(X)$	semi-Urysohn [1]
$PO(X)$	$PO(X)$	pre-Urysohn [16]

Every pairwise Urysohn space is also a pairwise Hausdorff space. The converse does not always hold. This follows from the following example.

EXAMPLE 2.19. Let $X = \{a, b, c, d, e\}$. Let us consider $\mu = \mu_1 = \mu_2 = \{\phi, \{a, b\}, \{c, d\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, c, e\}, \{b, d, e\}, \{b, c, d, e\}, \{a, c, d, e\}, \{a, b, d, e\}, \{a, b, c, e\}, \{a, b, c, d\}, X\}$. Then for every pair of distinct x, y there exist disjoint μ -open set U, V containing x, y respectively. i.e. X is pairwise Hausdorff. But if we take a and c then there exist no pair of μ -open set U, V containing x, y respectively s.t. $c_{\mu_2}U \cap c_{\mu_1}V = \phi$. i.e. X is not pairwise Urysohn.

DEFINITION 2.20. Let (X, μ_1, μ_2) be a bi-GTS. Then for any subset A of X we define,

$$P(A) = (\cap\{\gamma_{\mu_1, \mu_2}(\gamma_{\mu_1, \mu_2}U) : A \subseteq U \in \mu_2\}) \cup (\cap\{\gamma_{\mu_2, \mu_1}(\gamma_{\mu_2, \mu_1}U) : A \subseteq U \in \mu_1\}).$$

LEMMA 2.21. For any point x in a bi-GTS (X, μ_1, μ_2) , $p_1[(X \times \{x\}) \cap \gamma_{\nu_i, \nu_j}\Delta] = p_2[(\{x\} \times X) \cap \gamma_{\nu_j, \nu_i}\Delta]$; $i, j = 1, 2$ ($i \neq j$).

Proof. Let $y \notin L.H.S.$ This implies that $(y, x) \notin \gamma_{\nu_i, \nu_j}\Delta$. So there exists $V \in \mu_i(y)$ and $U \in \mu_j(x)$ such that $c_{\nu_j}(V \times U) \cap \Delta = \phi$ i.e. $(c_{\mu_j}V \times c_{\mu_i}U) \cap \Delta = \phi$ which gives $c_{\mu_j}V \cap c_{\mu_i}U = \phi$. Thus we have $(c_{\mu_i}U \times c_{\mu_j}V) \cap \Delta = \phi$ i.e. $c_{\nu_i}(U \times V) \cap \Delta = \phi$ which gives $(x, y) \notin \gamma_{\mu_j, \mu_i}\Delta$ i.e. $y \notin R.H.S.$

By reversing the above arguments we can similarly show that $y \notin R.H.S.$ implies $y \notin L.H.S.$ \square

LEMMA 2.22. If (X, μ_1, μ_2) is a bi-GTS and $x \in X$, then

$$\begin{aligned} P(\{x\}) &= p_1[(X \times \{x\}) \cap \gamma_{\nu_i, \nu_j}\Delta] \cup p_2[(\{x\} \times X) \cap \gamma_{\nu_i, \nu_j}\Delta] \\ &= p_1[(X \times \{x\}) \cap \gamma_{\nu_1, \nu_2}\Delta] \cup p_1[(X \times \{x\}) \cap \gamma_{\nu_2, \nu_1}\Delta] \\ &= p_2[(\{x\} \times X) \cap \gamma_{\nu_1, \nu_2}\Delta] \cup p_2[(\{x\} \times X) \cap \gamma_{\nu_2, \nu_1}\Delta] \end{aligned}$$

Proof. In view of Lemma 2.21 it is sufficies to show that $P(\{x\}) = p_1[(X \times \{x\}) \cap \gamma_{\nu_1, \nu_2}\Delta] \cup p_1[(X \times \{x\}) \cap \gamma_{\nu_2, \nu_1}\Delta]$. Now, if $y \notin P(\{x\})$ then there exist $U_2 \in \mu_2(x)$ and $U_1 \in \mu_1(x)$ such that $y \notin \gamma_{\mu_1, \mu_2}(\gamma_{\mu_1, \mu_2}U_2)$ and $y \notin \gamma_{\mu_2, \mu_1}(\gamma_{\mu_2, \mu_1}U_1)$. Consequently we have $V_1 \in \mu_1(y)$ and $V_2 \in \mu_2(y)$ such that $c_{\mu_2}V_1 \cap \gamma_{\mu_1, \mu_2}U_2 = \phi = c_{\mu_1}V_2 \cap \gamma_{\mu_2, \mu_1}U_1$ i.e. $c_{\mu_2}V_1 \cap c_{\mu_1}U_2 = \phi = c_{\mu_1}V_2 \cap c_{\mu_2}U_1$ (by Theorem 1.4). Then $(c_{\mu_2}V_1 \times c_{\mu_1}U_2) \cap \Delta = \phi = (c_{\mu_1}V_2 \times c_{\mu_2}U_1) \cap \Delta$ i.e. $c_{\nu_2}(V_1 \times U_2) \cap \Delta = \phi = c_{\nu_1}(V_2 \times U_1) \cap \Delta$ which gives $(y, x) \notin \gamma_{\nu_1, \nu_2}\Delta$ and $(y, x) \notin \gamma_{\nu_2, \nu_1}\Delta$. Hence $y \notin p_1[(X \times \{x\}) \cap \gamma_{\nu_1, \nu_2}\Delta]$ and $y \notin p_1[(X \times \{x\}) \cap \gamma_{\nu_2, \nu_1}\Delta]$ so that $y \notin R.H.S.$ By reversing the above argument we can similarly show that $y \notin R.H.S.$ implies $y \notin L.H.S.$ \square

THEOREM 2.23. *For a bi-GTS (X, μ_1, μ_2) the following are equivalent:*

- (1) X is pairwise Urysohn.
- (2) For each $x \in X, \{x\} = P(\{x\})$.
- (3) $\Delta = (\gamma_{\nu_1, \nu_2} \Delta) \cup (\gamma_{\nu_2, \nu_1} \Delta)$.

Proof.

(1) \Rightarrow (2): Let $x \in X$ and y be any point of X with $y \neq x$. Then there exist $U \in \mu_1(x)$ and $V \in \mu_2(y)$ such that $c_{\mu_2}U \cap c_{\mu_1}V = \phi$ and so we have $\gamma_{\mu_2, \mu_1}U \cap c_{\mu_1}V = \phi$ (by Theorem 1.4) so that $y \notin \gamma_{\mu_2, \mu_1}(\gamma_{\mu_2, \mu_1}U)$. Similarly we can find $W \in \mu_2(x)$ such that $y \notin \gamma_{\mu_1, \mu_2}(\gamma_{\mu_1, \mu_2}W)$. Hence we get (2).

(2) \Rightarrow (3) : We have,

$$\begin{aligned} (x, y) \notin \Delta &\Leftrightarrow y \notin P(\{x\}) \\ &\Leftrightarrow y \notin p_2[(\{x\} \times X) \cap \gamma_{\nu_1, \nu_2} \Delta] \cup p_2[(\{x\} \times X) \cap \gamma_{\nu_2, \nu_1} \Delta] \\ &\Leftrightarrow (x, y) \notin \gamma_{\nu_1, \nu_2} \Delta \text{ and } (x, y) \notin \gamma_{\nu_2, \nu_1} \Delta \end{aligned}$$

Thus (3) follows.

(3) \Rightarrow (1): Let $x, y \in X$ such that $x \neq y$. Since $(x, y) \notin \Delta, (x, y) \notin \gamma_{\nu_1, \nu_2} \Delta$ so that $c_{\nu_2}(U_1 \times V_2) \cap \Delta = \phi$ for some $U_1 \in \mu_1(x)$ and $V_2 \in \mu_2(y)$. Then $(c_{\mu_2}U_1 \times c_{\mu_1}V_2) \cap \Delta = \phi$ i.e. $c_{\mu_2}U_1 \cap c_{\mu_1}V_2 = \phi$, proving that X is pairwise Urysohn. \square

COROLLARY 2.24. *A bi-GTS (X, μ_1, μ_2) is pairwise Urysohn iff any one of the following conditions holds:*

- (1) For each $x \in X, \{x\} = \cap \{ \gamma_{\mu_1, \mu_2}(\gamma_{\mu_1, \mu_2}U) : U \in \mu_2(x) \}$.
- (2) For each $x \in X, \{x\} = \cap \{ \gamma_{\mu_2, \mu_1}(\gamma_{\mu_2, \mu_1}U) : U \in \mu_1(x) \}$.
- (3) $\Delta = \gamma_{\mu_1, \mu_2} \Delta$.
- (4) $\Delta = \gamma_{\mu_2, \mu_1} \Delta$.

DEFINITION 2.25. ([14]) Let (X, μ_1, μ_2) be a bi-GTS. Then X is said to be (μ_i, μ_j) -regular if for any $x \in X$ and any μ_i -closed set F not containing x , there exist $U \in \mu_i$ and $V \in \mu_j$ with $x \in U, F \subseteq V$ such that $U \cap V = \emptyset; i, j = 1, 2 (i \neq j)$.

If X is (μ_1, μ_2) and (μ_2, μ_1) regular then X is called pairwise regular.

μ_1	μ_2	(μ_1, μ_2) -regular
τ	τ	regular
Δ	τ	almost regular [17]
$SO(X)$	$SO(X)$	semi regular [6]
$PO(X)$	$PO(X)$	strong regular [13]

THEOREM 2.26. [14] A bi-GTS (X, μ_1, μ_2) is (μ_i, μ_j) -regular iff for each point $x \in X$ and each μ_i -open set G containing x , there is a μ_i -open set H containing x such that $c_{\mu_j}H \subseteq G$; $i, j = 1, 2$ ($i \neq j$).

THEOREM 2.27. A bi-GTS (X, μ_1, μ_2) is (μ_i, μ_j) -regular iff for any set A in X , $c_{\mu_i}A = \gamma_{\mu_i, \mu_j}A$; $i, j = 1, 2$ ($i \neq j$).

Proof. First suppose that X is (μ_i, μ_j) -regular. Obviously $c_{\mu_i}A \subseteq \gamma_{\mu_i, \mu_j}A$ for $A \subseteq X$. Now let $x \in \gamma_{\mu_i, \mu_j}A$ and U be any μ_i -open set containing x , then by Theorem 2.23 there exists a μ_i -open set V containing x such that $c_{\mu_j}V \subseteq U$. Now since $x \in \gamma_{\mu_i, \mu_j}A$, we get $c_{\mu_j}V \cap A \neq \phi$ and hence $U \cap A \neq \phi$. Thus $x \in c_{\mu_i}A$ and consequently, $\gamma_{\mu_i, \mu_j}A = c_{\mu_i}A$. Conversely, let $x \in X$ and U be a μ_i -open set containing x . Then $x \notin X \setminus U = c_{\mu_i}(X \setminus U) = \gamma_{\mu_i, \mu_j}(X \setminus U)$. Thus there exists a μ_i -open set V containing x such that $c_{\mu_j}V \cap (X \setminus U) = \phi$ i.e. $c_{\mu_j}V \subseteq U$ and hence X is (μ_i, μ_j) -regular. \square

COROLLARY 2.28. A Bi-BTS (X, μ_i, μ_j) is pairwise regular iff every μ_i -closed set is $\theta(\mu_i, \mu_j)$ -closed; $i, j = 1, 2$ ($i \neq j$).

DEFINITION 2.29. ([15]) A bi-GTS (X, μ_1, μ_2) is said to be (μ_i, μ_j) -almost regular if for each $x \in X$ and $r(\mu_i, \mu_j)$ -closed set F with $x \notin F$, there exist $U \in \mu_i$ and $V \in \mu_j$ such that $x \in U, F \subseteq V$ and $U \cap V = \phi$; $i, j = 1, 2$ ($i \neq j$).

X is called pairwise almost regular if it is both (μ_1, μ_2) -almost regular and (μ_2, μ_1) -almost regular.

It is easy to check that every pairwise regular space is also a pairwise almost regular space. But the converse is not so. This follows from the following example.

EXAMPLE 2.30. Let us consider the set $X = \{a, b, c, d\}$. Let $\mu_1 = \mu_2 = \mu = \{\phi, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then for the point a and the μ -closed set $F = \{b, c\}$ there exist no pair of μ -open sets U and V s.t. $a \in U, F \subseteq V$ and $U \cap V = \phi$. i.e. X is not pairwise regular. But the $r(\mu_i, \mu_j)$ -closed set in X are $\phi, \{a, b\}, \{c, d\}, \{b, d\}, \{a, c\}$. So for each $x \in X$ and $r(\mu_i, \mu_j)$ -closed set F with $x \notin F$, there exist $U \in \mu_i$ and $V \in \mu_j$ such that $x \in U, F \subseteq V$ and $U \cap V = \phi$. Hence X is pairwise almost regular.

THEOREM 2.31. ([15]) A bi-GTS (X, μ_1, μ_2) is (μ_i, μ_j) -almost regular iff for $x \in X$ and $r(\mu_i, \mu_j)$ -open set U containing x , there exists μ_i -open set V containing x such that $c_{\mu_j}V \subseteq U$; $i, j = 1, 2$ ($i \neq j$).

THEOREM 2.32. For a bi-GTS (X, μ_1, μ_2) the following are equivalent:

- (a) X is (μ_i, μ_j) -almost regular.
 (b) For any set $A \subseteq X$, $\gamma_{\mu_i, \mu_j} A = c_{\delta(\mu_i, \mu_j)} A$.
 (c) For any set $A \subseteq X$, $\gamma_{\mu_i, \mu_j}(\gamma_{\mu_i, \mu_j} A) = \gamma_{\mu_i, \mu_j} A$.
 (d) For any μ_j -open set A , $\gamma_{\mu_i, \mu_j}(\gamma_{\mu_i, \mu_j} A) = \gamma_{\mu_i, \mu_j} A$; $i, j = 1, 2$ ($i \neq j$).

Proof.

(a) \Rightarrow (b): It is always true that $c_{\delta(\mu_i, \mu_j)} A \subseteq \gamma_{\mu_i, \mu_j} A$. Let, $x \in \gamma_{\mu_i, \mu_j} A$ and $U \in \mu_i(x)$. Then by (a), there exists $V \in \mu_j(x)$ such that $c_{\mu_j} V \subseteq i_{\mu_i} c_{\mu_j} U$. Since $c_{\mu_j} V \cap A \neq \phi$, we have $(i_{\mu_i} c_{\mu_j} U) \cap A \neq \phi$ and thus $x \in c_{\delta(\mu_i, \mu_j)} A$.

(b) \Rightarrow (c): We have,

$$\begin{aligned} \gamma_{\mu_i, \mu_j}(\gamma_{\mu_i, \mu_j} A) &= \gamma_{\mu_i, \mu_j}(c_{\delta(\mu_i, \mu_j)} A) = c_{\delta(\mu_i, \mu_j)}(c_{\delta(\mu_i, \mu_j)} A) \\ &= c_{\delta(\mu_i, \mu_j)} A = \gamma_{\mu_i, \mu_j} A. \end{aligned}$$

(c) \Rightarrow (d): Straightforward.

(d) \Rightarrow (a): Let F be any $r(\mu_i, \mu_j)$ -open set in X and $p \in F$. Now $A = X \setminus F$ is an $r(\mu_i, \mu_j)$ -closed set and then $A = c_{\mu_i}(i_{\mu_j} A)$. Put $B = i_{\mu_j} A$. Then $\gamma_{\mu_i, \mu_j} A = \gamma_{\mu_i, \mu_j}(c_{\mu_i} B) = \gamma_{\mu_i, \mu_j}(\gamma_{\mu_i, \mu_j} B) = \gamma_{\mu_i, \mu_j} B = c_{\mu_i} B = A$. Then $p \notin \gamma_{\mu_i, \mu_j} A$ and hence there exist $G \in \mu_i(p)$ such that $c_{\mu_j} G \cap A = \phi$ i.e. $c_{\mu_j} G \subseteq X \setminus A = F$. Hence X is (μ_i, μ_j) -almost regular. \square

3. Separation axioms via generalized cluster sets and graph of a function

This section is devoted to establish necessary and sufficient conditions for separation properties of a bi-GTS via generalized cluster sets and graph of a function. We begin with a few useful lemmas and already known definitions.

LEMMA 3.1. Let f be a function from a set X to a set Y . Then for any $A \subseteq X$ and any $B \subseteq Y$, $f(A) \cap B = \{y \in Y : (x, y) \in ((A \times B) \cap G(f)), \text{ for some } x \in X\}$.

LEMMA 3.2. Let (X, μ_1, μ_2) and (Y, η_1, η_2) be bi-GTS. Then $\gamma_{\nu_i, \nu_j}\{(x, y)\} = \gamma_{\mu_i, \mu_j}\{x\} \times \gamma_{\eta_j, \eta_i}\{y\}$, for any $(x, y) \in X \times Y$.

Proof. Let $(a, b) \in \gamma_{\nu_i, \nu_j}\{(x, y)\}$ and $U \in \mu_i(a)$, $V \in \eta_j(b)$. Then $(x, y) \in c_{\nu_j}(U \times V) \Rightarrow (x, y) \in c_{\mu_j} U \times c_{\eta_i} V \Rightarrow x \in c_{\mu_j} U$ and $y \in c_{\eta_i} V$. Hence $a \in \gamma_{\mu_i, \mu_j}\{x\}$ and $b \in \gamma_{\eta_j, \eta_i}\{y\}$. This shows that $(a, b) \in \gamma_{\mu_i, \mu_j}\{x\} \times \gamma_{\eta_j, \eta_i}\{y\}$. Then $\gamma_{\nu_i, \nu_j}\{(x, y)\} \subseteq \gamma_{\mu_i, \mu_j}\{x\} \times \gamma_{\eta_j, \eta_i}\{y\}$. Reversing the argument we get the reverse inclusion. Hence $\gamma_{\nu_i, \nu_j}\{(x, y)\} = \gamma_{\mu_i, \mu_j}\{x\} \times \gamma_{\eta_j, \eta_i}\{y\}$ for any $(x, y) \in X \times Y$. \square

DEFINITION 3.3. ([2]) Let $f : (X, \mu_1, \mu_2) \rightarrow (Y, \eta_1, \eta_2)$ be a function. Then for any $x \in X$, then the generalized cluster set of f at any point x is given by $\mathcal{G}_{ij}^{kl}(f, x) = \cap\{\gamma_{\eta_k, \eta_l} f(c_{\mu_j} U) : U \in \mu_i(x)\}$ $i, j, k, l = 1, 2$ ($i \neq j$ and $k \neq l$).

LEMMA 3.4. Let $f : (X, \mu_1, \mu_2) \rightarrow (Y, \eta_1, \eta_2)$ be a function and $x \in X$. Then

- (1) $p_2((\{x\} \times Y) \cap \gamma_{\nu_i, \nu_j} G(f)) = \mathcal{G}_{ij}^{ji}(f, x)$.
- (2) $p_2((\{x\} \times Y) \cap k_{\nu_i} G(f)) = k_{\eta_j}(f(c_{\mu_i}\{x\}))$.
- (3) $p_2((\{x\} \times Y) \cap \gamma'_{\nu_i, \nu_j} G(f)) = \gamma'_{\eta_j, \eta_i}(f(\gamma_{\mu_i, \mu_j}\{x\}))$.

Proof.

- (1) Let $y \in \mathcal{G}_{ij}^{ji}(f, x)$ and $U \in \mu_i(x), V \in \eta_j(y)$. Then $y \in \gamma_{\eta_j, \eta_i} f(c_{\mu_j} U)$ and so $c_{\eta_i} V \cap f(c_{\mu_j} U) \neq \phi$ i.e. $(c_{\mu_j} U \times c_{\eta_i} V) \cap G(f) \neq \phi$ i.e. $c_{\nu_j}(U \times V) \cap G(f) \neq \phi$. This shows that $(x, y) \in \gamma_{\nu_i, \nu_j} G(f)$; so that $y \in p_2((\{x\} \times Y) \cap \gamma_{\nu_i, \nu_j} G(f))$. Reversing the step we get the reverse inclusion. Hence $p_2((\{x\} \times Y) \cap \gamma_{\nu_i, \nu_j} G(f)) = \mathcal{G}_{ij}^{ji}(f, x)$.
- (2) Let $y \in L.H.S.$ Then $(x, y) \in k_{\nu_i} G(f)$ i.e. $c_{\nu_i}\{x, y\} \cap G(f) \neq \phi$, which gives $(c_{\mu_i}\{x\} \times c_{\eta_j}\{y\}) \cap G(f) \neq \phi$ so $f(c_{\mu_i}\{x\}) \cap c_{\eta_j}\{y\} \neq \phi$ and hence $y \in k_{\eta_j}(f(c_{\mu_i}\{x\}))$. i.e. $y \in R.H.S.$ Then $L.H.S. \subseteq R.H.S.$
- (3) Let $y \in p_2((\{x\} \times Y) \cap \gamma'_{\nu_i, \nu_j} G(f))$. Then $(x, y) \in \gamma'_{\nu_i, \nu_j} G(f)$ i.e. $\gamma_{\nu_i, \nu_j}\{x, y\} \cap G(f) \neq \phi$. So by lemma 3.2 $(\gamma_{\mu_i, \mu_j}\{x\} \times \gamma_{\eta_j, \eta_i}\{y\}) \cap G(f) \neq \phi$. Then by lemma 3.1 $f(\gamma_{\mu_i, \mu_j}\{x\}) \cap \gamma_{\eta_j, \eta_i}\{y\} \neq \phi$, which gives $y \in \gamma'_{\eta_j, \eta_i}(f(\gamma_{\mu_i, \mu_j}\{x\}))$. Hence $p_2((\{x\} \times Y) \cap \gamma'_{\nu_i, \nu_j} G(f)) \subseteq \gamma'_{\eta_j, \eta_i}(f(\gamma_{\mu_i, \mu_j}\{x\}))$. Reversal of above arguments yields the inclusion the other way round. \square

COROLLARY 3.5. If $f : (X, \mu_1, \mu_2) \rightarrow (Y, \eta_1, \eta_2)$ is a function then the following are equivalent:

- (1) f has a $\theta(\nu_i, \nu_j)$ -closed graph.
- (2) $\{f(x)\} = p_2((\{x\} \times Y) \cap \gamma_{\nu_i, \nu_j} G(f))$, for each $x \in X$.
- (3) $\mathcal{G}_{ij}^{ji}(f, x) = \{f(x)\}$, for each $x \in X$.

DEFINITION 3.6. ([2]) Let (X, μ_1, μ_2) and (Y, η_1, η_2) be two bi-GTS. Then $f : (X, \mu_1, \mu_2) \rightarrow (Y, \eta_1, \eta_2)$ is said to be $(\mu_i \mu_j, \eta_k)$ -continuous at $x \in X$ if for each $V \in \eta_k(f(x))$, there exists $U \in \mu_i(x)$ such that $f(c_{\mu_j} U) \subseteq V$. $i, j, k = 1, 2$ ($i \neq j$).

If f is $(\mu_i \mu_j, \eta_k)$ -continuous at each $x \in X$ then f is called $(\mu_i \mu_j, \eta_k)$ -continuous on X .

If f is both $(\mu_i \mu_j, \eta_1)$ and $(\mu_i \mu_j, \eta_2)$ -continuous then f is called pairwise $(\mu_i \mu_j, \eta_k)$ -continuous.

LEMMA 3.7. If $f : (X, \mu_1, \mu_2) \rightarrow (Y, \eta_1, \eta_2)$ is a function such that f is $(\mu_i \mu_j, \eta_i)$ continuous then, $\mathcal{G}_{ij}^{ji}(f, x) = \gamma_{\eta_j, \eta_i} \{f(x)\}$, for each $x \in X$.

Proof. Let $V \in \eta_i$ such that $f(x) \in V$. Since f is $(\mu_i \mu_j, \eta_i)$ continuous, there exists $U \in \mu_i(x)$ such that $f(c_{\mu_j} U) \subseteq V$. Then $\mathcal{G}_{ij}^{ji}(f, x) \subseteq \gamma_{\eta_j, \eta_i} f(c_{\mu_j} U) \subseteq \gamma_{\eta_j, \eta_i} V = c_{\eta_j} V$ (by Theorem 1.4). Thus $\mathcal{G}_{ij}^{ji}(f, x) \subseteq c_{\eta_j} V$ for all $V \in \eta_i$ with $f(x) \in V$. By Theorem 1.5 $\mathcal{G}_{ij}^{ji}(f, x) \subseteq \gamma_{\eta_j, \eta_i} \{f(x)\}$. On the other hand, $\gamma_{\eta_j, \eta_i} \{f(x)\} \subseteq \mathcal{G}_{ij}^{ji}(f, x)$ is obvious. Hence the lemma. \square

COROLLARY 3.8. A function $f : (X, \mu_1, \mu_2) \rightarrow (Y, \eta_1, \eta_2)$ has the property that f is $(\mu_i \mu_j, \eta_i)$ continuous, has a $\theta(\nu_i, \nu_j)$ -closed graph iff it has $\theta(\eta_j, \eta_i)$ -closed point images.

LEMMA 3.9. A function $f : (X, \mu_1, \mu_2) \rightarrow (Y, \eta_1, \eta_2)$ is $(\mu_i \mu_j, \eta_k)$ continuous iff $f(\gamma_{\mu_i, \mu_j} A) \subseteq c_{\eta_k} f(A)$; $i, j, k = 1, 2$ ($i \neq j$).

Proof. Let f be a $(\mu_i \mu_j, \eta_k)$ continuous and $y \in f(\gamma_{\mu_i, \mu_j} A)$. There exists $x \in X$ such that $x \in \gamma_{\mu_i, \mu_j} A$ and $f(x) = y$. Let $V \in \eta_k(f(x))$. Then there exists $U \in \mu_i(x)$ such that $f(c_{\mu_j} U) \subseteq V$. Again since $x \in \gamma_{\mu_i, \mu_j} A$ we have $c_{\mu_j} U \cap A \neq \phi$ and so $f(c_{\mu_j} U) \cap f(A) \neq \phi$ i.e $V \cap f(A) \neq \phi$ i.e $f(x) \in c_{\eta_k} f(A)$ and hence $y \in c_{\eta_k} f(A)$.

Conversely, let $x \in X$ be arbitrary and $V \in \eta_k(f(x))$. Then $f(x) \notin c_{\eta_k}(Y \setminus V)$ and so $f(x) \notin c_{\eta_k}(ff^{-1}(Y \setminus V))$. By the hypothesis $f(x) \notin f(\gamma_{\mu_i, \mu_j}(f^{-1}(Y \setminus V)))$, so that $x \notin \gamma_{\mu_i, \mu_j}(X \setminus (f^{-1}V))$. Thus there exists $U \in \mu_i(x)$ such that $c_{\mu_j} U \subseteq f^{-1}V$ i.e. $f(c_{\mu_j} U) \subseteq V$. Hence f is $(\mu_i \mu_j, \eta_k)$ continuous. \square

THEOREM 3.10. If $f : (X, \mu_1, \mu_2) \rightarrow (Y, \eta_1, \eta_2)$ is pairwise $(\mu_i \mu_j, \eta_k)$ continuous and (Y, η_1, η_2) is pairwise R_1 , then $\gamma_{\nu_i, \nu_j} G(f) = k_{\nu_j} G(f)$.

Proof. Since f is pairwise $(\mu_i \mu_j, \eta_k)$ continuous and (Y, η_1, η_2) is pairwise R_1 , by Lemma 3.7, Theorem 2.11 and Lemma 3.9 we have, $\mathcal{G}_{ij}^{ji}(f, x) = \gamma_{\eta_j, \eta_i} \{f(x)\} = k_{\eta_i} \{f(x)\}$ and $f(\gamma_{\mu_i, \mu_j} \{x\}) \subseteq c_{\eta_j} \{f(x)\}$. So $k_{\eta_i} \{f(x)\} \subseteq k_{\eta_i} f(c_{\mu_j} \{x\}) \subseteq k_{\eta_i} f(\gamma_{\mu_j, \mu_i} \{x\}) \subseteq k_{\eta_i}(c_{\eta_j} \{f(x)\})$. Again by theorem 2.11 $ji-ck\{f(x)\} = \gamma_{\eta_j, \eta_i} \{f(x)\} = c_{\eta_j} \{f(x)\}$. So by Lemma 2.3 $k_{\eta_i} \{f(x)\} = k_{\eta_i}(ji-ck\{f(x)\}) = k_{\eta_i}(c_{\eta_j} \{f(x)\})$. Hence $k_{\eta_i} \{f(x)\} = k_{\eta_i} f(c_{\mu_j} \{x\})$. i.e. $\mathcal{G}_{ij}^{ji}(f, x) = k_{\eta_i} f(c_{\mu_j} \{x\})$. It follows from lemma 3.4 $\gamma_{\nu_i, \nu_j} G(f) = k_{\nu_j} G(f)$. \square

THEOREM 3.11. If $f : (X, \mu_1, \mu_2) \rightarrow (Y, \eta_1, \eta_2)$ is $(\mu_i \mu_j, \eta_i)$ continuous and (Y, η_1, η_2) is pairwise Hausdorff, then $\gamma_{\nu_i, \nu_j} G(f) = \gamma'_{\nu_i, \nu_j} G(f)$.

Proof. Since f is $(\mu_i\mu_j, \eta_i)$ continuous and (Y, η_1, η_2) is pairwise Hausdorff, by Lemma 3.7, Corollary 2.17 and Lemma 3.9 we have $\mathcal{G}_{ij}^{ji}(f, x) = \gamma_{\eta_j, \eta_i}\{f(x)\} = \{f(x)\}$ and $f(\gamma_{\mu_i, \mu_j}\{x\}) \subseteq c_{\eta_i}\{f(x)\}$. Again using Theorem 2.16, Lemma 3.9 and Corollary 2.17 we have

$$\begin{aligned} \{f(x)\} &\subseteq \gamma'_{\eta_j, \eta_i}(f(\gamma_{\mu_i, \mu_j}\{x\})) \\ &= f(\gamma_{\mu_i, \mu_j}\{x\}) \subseteq c_{\eta_i}\{f(x)\} \subseteq \gamma_{\eta_i, \eta_j}\{f(x)\} = \{f(x)\}. \end{aligned}$$

Then $\gamma'_{\eta_j, \eta_i}(f(\gamma_{\mu_i, \mu_j}\{x\})) = \{f(x)\} = \mathcal{G}_{ij}^{ji}(f, x)$. Hence by Lemma 3.4 $\gamma_{\nu_i, \nu_j}G(f) = \gamma'_{\nu_i, \nu_j}G(f)$. \square

THEOREM 3.12. *If a bi-GTS (Y, η_1, η_2) is pairwise R_1 , then for any bi-GTS (X, μ_1, μ_2) and every $(\mu_i\mu_j, \eta_i)$ continuous function $f : (X, \mu_1, \mu_2) \rightarrow (Y, \eta_1, \eta_2)$ with η_j -closed point image, f has a $\theta(\nu_i, \nu_j)$ -closed graph.*

Proof. Let f be a $(\mu_i\mu_j, \eta_i)$ continuous function from a bi-GTS (X, μ_1, μ_2) to (Y, η_1, η_2) with η_i closed point image and let Y be a pairwise R_1 . Then for each $x \in X$, $\{f(x)\} = c_{\eta_j}\{f(x)\} = \gamma_{\eta_j, \eta_i}\{f(x)\}$ (by Theorem 2.11) $= \mathcal{G}_{ij}^{ji}(f, x)$ (by Lemma 3.7). Then by the Corollary 3.5 f has a $\theta(\nu_i, \nu_j)$ closed graph. \square

THEOREM 3.13. *If a bi-GTS (Y, η_1, η_2) is pairwise Hausdorff then every $(\mu_i\mu_j, \eta_i)$ continuous function f from any (X, μ_1, μ_2) to (Y, η_1, η_2) has an $\theta(\nu_i, \nu_j)$ -closed graph.*

Proof. It follows from Lemma 3.7, Corollary 2.17 and Corollary 3.5. \square

DEFINITION 3.14. A multifunction $F : (X, \mu_1, \mu_2) \rightarrow (Y, \eta_1, \eta_2)$ is called $(\mu_i\mu_j, \eta_k)$ continuous at a point x of X if for each η_k open set W in Y such that $F(x) \subseteq W$, there is a $V \in \mu_i(x)$ satisfying $F(c_{\mu_j}V) \subseteq W$, where $F(V) = \cup\{F(y) : y \in V\}$; F is $(\mu_i\mu_j, \eta_k)$ continuous if F is so at each $x \in X$. $i, j, k = 1, 2$ ($i \neq j$).

THEOREM 3.15. *If a bi-GTS (Y, η_1, η_2) is pairwise regular. Then for each $x \in X$ and each $(\mu_i\mu_j, \eta_i)$ continuous multifunction F from any bi-GTS (X, μ_1, μ_2) to (Y, η_1, η_2) , $\mathcal{G}_{ij}^{ji}(F, x) = c_{\eta_j}F(x)$ for $i, j = 1, 2$ ($i \neq j$), where $\mathcal{G}_{ij}^{ji}(F, x) = \cap\{\gamma_{\eta_j, \eta_i}F(c_{\mu_j}U) : U \in \mu_i(x)\}$.*

Proof. Let (Y, η_1, η_2) be pairwise regular. Now obviously $c_{\eta_j}F(x) \subseteq \mathcal{G}_{ij}^{ji}(F, x)$. On the other hand, if $x \in X$ and $W \in \eta_i$ such that $F(x) \subseteq W$, then by $(\mu_i\mu_j, \eta_i)$ continuity of F there exists $V \in \mu_i(x)$ such that $F(c_{\mu_j}V) \subseteq W$. So, $\mathcal{G}_{ij}^{ji}(F, x) = \cap\{\gamma_{\eta_j, \eta_i}F(c_{\mu_j}V) : V \in \mu_i(x)\} \subseteq$

$\cap\{\gamma_{\eta_j, \eta_i} W : F(x) \subseteq W \in \eta_i\} = \cap\{c_{\eta_j} W : F(x) \subseteq W \in \eta_i\}$. It suffices to show that $\cap\{c_{\eta_j} W : F(x) \subseteq W \in \eta_i\} = c_{\eta_j} F(x)$. In fact $c_{\eta_j} F(x) \subseteq \cap\{c_{\eta_j} W : F(x) \subseteq W \in \eta_i\}$ is obvious. Now let $y \in \cap\{c_{\eta_j} W : F(x) \subseteq W \in \eta_i\}$ and $y \notin c_{\eta_j} F(x)$. Since Y is pairwise regular, there exist $U' \in \eta_j$ and $V' \in \eta_i$ with $y \in U'$, $c_{\eta_j} F(x) \subseteq V'$ and $U' \cap V' = \phi$. But since $F(x) \subseteq V' \in \eta_i$ we have $y \in c_{\eta_j} V'$, which contradicts $U' \cap V' = \phi$. Hence $\mathcal{G}_{i,j}^{j,i}(F, x) = c_{\eta_j} F(x)$. \square

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Department of Mathematics
West Bengal State University
Kolkata-700026, India
E-mail: atasi@hotmail.com

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Department of Mathematics
West Bengal State University
Kolkata-700026, India
E-mail: r.bhowmick88@gmail.com